# Countability of the Real Numbers

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**Abstract**

The proofs that the real numbers are denumerable will be shown, i.e., that there exists one-to-one correspondence between the natural numbers $N$ and the real numbers $\Re$. The general element of the sequence that contains all real numbers will be explicitly specified, and the first few elements of the sequence will be written. Remarks on the Cantor's nondenumerability proofs of 1873 and 1891 that the real numbers are noncountable will be given.

*Key words:* denumerability, real numbers, countability, cardinal numbers
*MSC:* 11B05

## 1 Introduction

The first proof that it is impossible to establish a one-to-one correspondence between the natural numbers $N$ and the real numbers $\Re$ is older than a century. In December 1873 Cantor first proved non-denumerability of continuum and that first proof proceeded as follows[1,2,3,4]: Find a closed interval $I_0$ that fails to contain $r_0$ then find a closed subinterval $I_1$ of $I_0$ such that $I_1$ misses $r_1$ continue in this manner, obtaining an infinite nested sequence of closed intervals, $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$, that eventually excludes every one of the $r_n$; now let $d$ be a point lying in the intersection of all the Ia's; $d$ is a real number different from all of the $r_n$.

This proof that no denumerable sequence of elements of an interval [a,b] can contain all elements of [a,b] often is overlooked in favor of the 1891 diagonal argument[5], when reference is made to Cantor's proving the nondenumerability of the continuum. Cantor himself repeated this proof with some

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modifications[2,3,6,7,8,9,10,11,12,13,14] from 1874 to 1897, and today we have even more variations of this proof given by other authors. However, we have to note that they are in nuce similar; all of them include same modification of the Cantor's idea to derive a contradiction by defining in terms which cannot possibly be in the assumed denumerable sequence. So, in principle, all these proofs do not represent a significant change from Cantor's original idea and we can take them to be the same as the Cantor's proofs.

For the reason of clarity, we will not discuss objections to these proofs that have been raised earlier[15,16,17,18,19,20,21] or the legitimacy of these proofs from intuitionistic points of view [22] and their nonconstructive parts, namely appeal to the Bolzano-Weierstrass theorem[23] and inclusion of impredicative methods[24]. We will focus to show what is in principle wrong with the general idea of Cantor's proofs and consequently all other proofs related to the statement that the set of real numbers is not denumerable.

The main part of the paper is devoted to show that the real numbers are denumerable. The explicit denumerable sequence that contains all real numbers will be given. The general element that generates the sequence will be written as well as the first a few elements of that sequence. That there is one-to-one correspondence between the real numbers and the elements of the explicitly written sequence will be proven by the three independent proofs.

## 2 Profs of the denumerability of the real numbers

**Theorem 1**

The real numbers $\Re$ are denumerable; it is possible to establish a one-to-one correspondence between the natural numbers $N$ and the real numbers $\Re$. In other words, the cardinal number $c$ of the set of real numbers is equal to the cardinal number $\aleph_0$ of the set of natural numbers. The general element of the sequence that generates all elements of the set $\Re$ is as follows:

$$a_1{}^{a_2{}^{a_3{}^{\cdot^{\cdot^{\cdot^{a_n}}}}}} \tag{1}$$

where in (1) each element $a_i$ of bases and exponents has the following form:

$$a_i = (\frac{m_{i1}}{n_{i1}})^{[(\frac{m_{i2}}{n_{i2}})^{(\frac{m_{i3}}{n_{i3}})}]} \tag{2}$$

where $m_{ij}, n_{ij} \in N, i = 1,2,3,...n, j = 1,2,3.$

With the general element (1) it is possible to express each of the real numbers, and to generate the sequence which contains all real numbers. That can be done by writing (1) for all possible combinations of arguments, with the sum of all bases and exponents equal to 2,3, 4,... and so on. To do that in a way that will provide a one-to one correspondence between such produced set and the set of natural numbers $N$ all elements obtained by (1) can be for example arranged in the following way: a) For the fixed sum of bases and exponents write all possible fractions $\frac{m_{11}}{n_{11}}$ of $a_1$ with lower denominator coming first. b) After that for the same fixed sum as before, write all elements $a_1 = (\frac{m_{11}}{n_{11}})^{(\frac{m_{12}}{n_{12}})}$, if it is possible to create such elements for that sum. Doing that, write first all possible combinations of $\frac{m_{11}}{n_{11}}$ for fixed $\frac{m_{12}}{n_{12}}$ and only after that change $\frac{m_{12}}{n_{12}}$ if it exists for that sum. During that elements with lower denominator $n_{11}$ and $n_{12}$ will be written first again. c) If it is possible for that specific fixed sum of bases and exponents, following the same rules a) and b), continue by writing elements of shape $a_i = (\frac{m_{i1}}{n_{i1}})^{[(\frac{m_{i2}}{n_{i2}})^{(\frac{m_{i3}}{n_{i3}})}]}$. Again first change will be done in $\frac{m_{11}}{n_{11}}$ after that change of $\frac{m_{12}}{n_{12}}$ and at lastly the change of $\frac{m_{13}}{n_{13}}$. d) When all possible combinations of $a_1$ are written for the fixed sum of bases and exponents, continue with increasing the number of exponents, if it is possible for that sum, and continue by writing all combinations that correspond to ${a_1}^{a_2}, {a_1}^{{a_2}^{a_3}}, {a_1}^{{a_2}^{{a_3}^{a_4}}}$, ... and so on. During that, first change exponents $a_i$ with lower index. e) When all possible increases of exponents are done and all possible combinations for the fixed sum are written, increase the value of the sum and repeat procedures a) through e). In addition all elements that appear again will not be written down. First few elements of this sequence are as follows:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \frac{1}{5}, (\frac{2}{1})^{\frac{1}{2}}, (\frac{1}{2})^{\frac{1}{2}}, \frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, (\frac{3}{1})^{\frac{2}{1}}, (\frac{1}{3})^{\frac{2}{1}}, (\frac{3}{1})^{\frac{1}{2}}, (\frac{1}{3})^{\frac{1}{2}},$$

$$(\frac{2}{1})^{\frac{3}{1}}, (\frac{1}{2})^{\frac{3}{1}}, (\frac{2}{1})^{\frac{1}{3}}, (\frac{1}{2})^{\frac{1}{3}}, \frac{7}{1}, \frac{5}{3}, \frac{3}{5}, \frac{1}{7}, (\frac{4}{1})^{\frac{2}{1}}, (\frac{3}{2})^{\frac{2}{1}}, (\frac{2}{3})^{\frac{2}{1}}, (\frac{1}{4})^{\frac{2}{1}}, (\frac{3}{2})^{\frac{1}{2}}, (\frac{2}{3})^{\frac{1}{2}}, (\frac{3}{1})^{\frac{3}{1}}, (\frac{1}{3})^{\frac{3}{1}}, (\frac{3}{1})^{\frac{1}{3}}, (\frac{1}{3})^{\frac{1}{3}},$$

$$(\frac{2}{1})^{\frac{3}{2}}, (\frac{1}{2})^{\frac{3}{2}}, (\frac{2}{1})^{\frac{1}{4}}, (\frac{1}{2})^{\frac{1}{4}}, \frac{7}{2}, \frac{5}{4}, \frac{4}{5}, \frac{2}{7}, (\frac{5}{1})^{\frac{2}{1}}, (\frac{1}{5})^{\frac{2}{1}}, (\frac{5}{1})^{\frac{1}{2}}, (\frac{1}{5})^{\frac{1}{2}}, (\frac{4}{1})^{\frac{3}{1}}, (\frac{3}{2})^{\frac{3}{1}}, (\frac{2}{3})^{\frac{3}{1}}, (\frac{1}{4})^{\frac{3}{1}}, (\frac{4}{1})^{\frac{1}{3}},$$

$$(\frac{3}{2})^{\frac{1}{3}}, (\frac{2}{3})^{\frac{1}{3}}, (\frac{1}{4})^{\frac{1}{3}}, (\frac{3}{1})^{\frac{4}{1}}, (\frac{1}{3})^{\frac{4}{1}}, (\frac{3}{1})^{\frac{3}{2}}, (\frac{1}{3})^{\frac{3}{2}}, (\frac{3}{1})^{\frac{2}{3}}, (\frac{1}{3})^{\frac{2}{3}}, (\frac{3}{1})^{\frac{1}{4}}, (\frac{1}{3})^{\frac{1}{4}}, (\frac{2}{1})^{\frac{5}{1}}, (\frac{1}{2})^{\frac{5}{1}}, (\frac{2}{1})^{\frac{1}{5}}, (\frac{1}{2})^{\frac{1}{5}},$$

$$(\frac{2}{1})^{[(\frac{2}{1})^{\frac{1}{2}}]}, (\frac{1}{2})^{[(\frac{2}{1})^{\frac{1}{2}}]}, (\frac{2}{1})^{[(\frac{1}{2})^{\frac{1}{2}}]}, (\frac{1}{2})^{[(\frac{1}{2})^{\frac{1}{2}}]}, ... \tag{3}$$

**Proof of the theorem 1**

Note first that it is obvious that the sequence contains all rational and algebraic numbers, and that transcendental numbers are included also, as in the case of algebraic irrational exponents and algebraic bases [25-31], for instance, for

$$(\frac{2}{1})^{[(\frac{2}{1})^{\frac{1}{2}}]} = 2^{\sqrt{2}} \tag{4}$$

Exponents $a_i$ in the general element (1) can be either algebraic or transcendental, which depends on arguments $m_{i,j}, n_{i,j}$ of $a_i$. How $a_i$ has the shape (2) and the arguments of $a_i$ can be changed for an arbitrary small amount, it is obvious that $a_i$ can obtain a value in any chosen interval. Since the general elements of the sequence (1) have the form ${a_1}^{{a_2}^{{a_3}^{\cdot^{\cdot^{\cdot^{a_n}}}}}}$, hence exponential function is continuous, and because values of the arguments of (1), $a_1, a_2, a_3, ..., a_n$ can be chosen from any interval and can be changed independently one from another for an arbitrary small amount, it follows that expression (1) can obtain any arbitrary value. Therefore, with (1), in any arbitrary chosen interval one can generate infinitely many algebraic and transcendental numbers, which is actually the continuum [32]. That means that with (1) we can represent any real number, which proves the theorem 1.

Since this is an extremely important issue, we will give two additional completely independent proofs of the theorem. However, before that let us consider some properties of the sequence generated with (1).

We need to note that only with $a_1$, which also can be transcendental, such as in (4), it is not possible to express all numbers, for instance the number e, as it requires

$$e = \frac{m_1}{n_1}^{[\frac{m_2}{n_2}^{\frac{m_3}{n_3}}]} \tag{5}$$

that is

$$1 = \frac{m_2}{n_2}^{\frac{m_3}{n_3}} ln\frac{m_1}{n_1} \tag{6}$$

and this cannot be, because $ln\frac{m_1}{n_1}$ is always transcendental [31-33] for $m_1, n_1 \in N$. However, it is necessary to note that no reason exists that could prevent expressing any arbitrary number with ${a_1}^{a_2}$. Therefore it may already be possible by ${a_1}^{a_2}$ to express all real numbers, i.e. maybe it is not necessary to build numbers with more and more exponents, i.e. numbers of the shape

$$
{a_1}^{{a_2}^{a_3}}, {a_1}^{{a_2}^{{a_3}^{a_4}}}, ..., {a_1}^{{a_2}^{{a_3}^{\cdot^{\cdot^{\cdot^{a_n}}}}}}, ... \tag{7}
$$

However, for now that statement can not be established, because it is not possible for now to calculate ${a_1}^{a_2}$ for the general case when both $a_1$ and $a_2$ are transcendental [34], it is not even possible to calculate it for earlier simpler case (4).

We need also to note the following: the statement that ${a_1}^{a_2}$ has only $\aleph_0\aleph_0 = \aleph_0$ elements and that this is the reason why it cannot contain all real numbers, which we have ${\aleph_0}^{\aleph_0}$ is not a good argument, because $\aleph_0 = {\aleph_0}^{\aleph_0}$ if the set of real number is countable, as it is. Consequently the possibility that the set of all real numbers could be expressed by only ${a_1}^{a_2}$ must be kept open.

**Theorem 2**

The set of numbers, generated by general element (1) and procedure a) through e) given in theorem (1), does not have any gaps. At each cut of the set the first component of the cut has the last element, or the second component of the cut has the first element, or both of these cases occur.

**The proof of the theorem 2**

In the definition of the theorem the meaning of the cut is simply a rule for dividing a set into two non-empty parts $A$ and $B$ such that every element of $A$ precedes every element of $B$ while $A$ and $B$ together exhaust the set.

Let denote with $S$ the set generated by (1) followimng the procedure given in theorem 1. The set $S$ obviously has as a subset the set of algebraic numbers $R$. If the $A/B$ is the cut in $S$, where $A$ is the first component and $B$ the second component of the cut, then $(A \cap R)/(B \cap R)$ is the certain defined cut $k$. If the component $A$ of the cut $A/B$ has the last element, then the cut does not generate new elements, $k \in A$ and $k$ is the last element of $A$.

If the $k$ is not the last element in $A$ then exists

$$
k' \in A \quad such \quad that \quad k < k'. \tag{8}
$$

But it is not possible that $k' \in R$, since it requires that $k' \in B \cap R$ and consequently $k' \in B$, which is not possible because of (8), since $A$ and $B$, as components of the cut do not have the common elements. So, if $k' \notin R$, than it means that $k'$ is a given cut $C/D$, of the set R, created with a gap in the set $R$ and for that reason the first component $C$ of the cut $C/D$ does not have the greatest element. Since $k < k'$, it follows that $A \cap R \subset C$. If $k''$ is any element of the set $C \backslash A$ then $k < k''$ will require that $k'' \in B$. But because of the $k'' \leq k'$, $k' \in A$ it requires $k'' \in A$. Both relations $k'' \in A$ and $k'' \in B$ can not be satisfied, because $A$ and $B$ as the components of the cut in $S$ are disjunctive sets. So, if $k \in A$ then $k$ is the last element of the component $A$.

In the same way it can be proven that if the $k \in S \backslash A$, so $k \in B$, then the $k$ is the first element of the second component $B$.

This proves the theorem. It is important to note that in proving this theorem we used the following properties of the set $S$: a) that it has the dense subset of the algebraic numbers $R$ and b) the set $S$ is everywhere dense, consequently for any $C \backslash A$ the general element of the set, relation (1), will generate numbers $k'' \in C \backslash A$, which are required to be in both $A$ and $B$, which established the contradiction.

It is not necessary to note that this also proves that the set $S$ is equivalent to the set of all real numbers $\Re$, since the set does not have the first and the last element, it is dense, it does not have any gaps, and it is linear.

**Theorem 3**

The set $S$, generated by the general element (1) and procedure a) through e) given in theorem 1, is similar (isomorphic) to the set of all real numbers $\Re$.

**The proof of the theorem 3**

The set $\Re = (\Re :<)$ of all real numbers ordered by the magnitude of its elements has the following properties: a) it does not have the first and the last element, b) it is continuous in Dedekin's sense, and c) it is separable. Any other set with properties a) to c) is similar to the set of real numbers $\Re$. Let us now show that the set $S$ given with theorem 1 and general element (1) satisfies the conditions a) to c) and is similar to the set of real numbers $\Re$. The set $S$ obviously satisfies properties a) and c). It also satisfies property given by b), as it is proven by theorem 2. With that the theorem 3 is proven. However, to keep this proof independent from the theorem 2, we will now prove it without using that theorem.

Let us denote by $M_1 \subseteq \Re$ any countable part of set $\Re$, such that it satisfies

condition: d) each interval of $\Re$ contains at least one element of $M_1$ and each interval of $M_1$ contains at least one element from $\Re \backslash M_1$. It is obvious that the set $M_1$ with the property d) can be created, since between any two algebraic numbers exist at least one transcedental number and between any two transcedental numbers exist at least one algebraic number [35].

Let us denote by $M_2 \subseteq S$ any countable part of $S$, such that each interval of $S$ contains at least one element of $M_2$ and each interval of $M_2$ contains at least one element of $S \backslash M_2$. The set $M_2$ also can be created, since the set $S$ generated by (1) also obviously has the property d). The set $S$ has as a subset the set of algebraic numbers. Also between any arbitrary chosen pairs of algebraic numbers it is possible to create transcedental numbers by the general element of sequence (1), and between any pairs of transcedental numbers generated by (1) there are algebraic numbers generated by (1).

Further, the sets $M_1$ and $M_2$ satisfy the following: a) sets do not have the first or the last element, b) sets are dense, c) sets are countable.

The sets $M_1$ and $M_2$ are similar to the set of algebraic numbers. We will now show that any similarity

$$\varphi(x), (x \in M_1), \varphi(M_1) = M_2 \tag{9}$$

between $M_1$ and $M_2$ can be extended on the similarity between entire $\Re$ and $S$.

Let take $x \in \Re \backslash M_1$ then we have cut

$$M_1 = (-\infty, x)_{M_1} \cup (x, \infty)_{M_1} \tag{10}$$

in the set $M_1$, which because of the density of the set $M_1$ opens a gap in $M_1$ and an element $x \in \Re \backslash M_1$ fulfills that gap in $\Re$. The $x$ is actually the only element that is between summands (10).

The cut in the set $M_1$ by the similarity (9) makes cut

$$M_2 = \varphi(M_1) = \varphi(-\infty, x)_{M_1} \cup \varphi(x, \infty)_{M_1} \tag{11}$$

of the set $M_2$. Because of the similarity of the sets $M_1$ and $M_2$, the cut (11) creates the gap in $M_2$. In that gap, because of the property d) is the element of the set $S$, which is defined by the similarity between $M_1$ and $M_2$, and by the element $x \in \Re \backslash M_1$, i.e. by the $\varphi(x)$. By that the transformation $\varphi(x)$ is defined for each $x \in \Re$. Obviously $\varphi(\Re) = S$. With that the theorem is proven, the set $S$ is similar to the set $\Re$.

## 3 Remarks on the Cantor's proofs

The above proposed sequence that contains all real numbers, and established denumerability of the real numbers are obviously in contradiction with the Cantor's proofs of nondenumerability. It is not to us to find the errors in Cantor's proofs and all numerous variations of his proofs that currently exist. However, we will give remarks on the Cantor's two most quoted proofs, from 1873 and 1891.

**Theorem 4**

In the Cantor's 1873 proof of nondenumerability, Cantor stated that it is possible to create sequences of progression and regression of elements, which allow for any interval of real numbers $(\alpha ... \beta)$ to define, in the limit, a number $\eta \in (\alpha, \beta)$, which was not included in the sequence assumed to contains all real numbers. The existence of the limit $\eta$ does not lead to the conclusion that the number $\eta$ is not in the sequence assumed to countain all real numbers and that the set of all real numbers is not countable.

**Proof of the theorem 4**

Let us now look in Cantor's 1873 nondenumerability proof, which appeared in Crelle's Journal in January 1874.

Assuming that the real numbers are countable, it follows that they could be sequenced on an index of natural number $N$:

$$\omega_1, \omega_2, \omega_3, ..., \omega_\nu, ... \tag{12}$$

Cantor then stated that for any given interval $(\alpha ... \beta)$ he could show the existence of a number $\eta \in (\alpha, \beta)$ which is not included in the sequence (12).

Assuming $\alpha < \beta$, he picked the first two numbers from (12), which fell within the interval $(\alpha, \beta)$. Denoted $\alpha', \beta'$, respectively, these were used to constitute another interval $(\alpha' ... \beta')$. Proceeding analogously, Cantor provided a sequence of nested intervals, reaching $(\alpha^{(\nu)} ... \beta^{(\nu)})$, where $\alpha^{(\nu)}, \beta^{(\nu)}$ were the first two numbers from (12) lying within $(\alpha^{(\nu-1)} ... \beta^{(\nu-1)})$

If the number of intervals thus constructed were finite, then at most only one more element from (12) could lie in $(\alpha^{(\nu)}, \beta^{(\nu)})$. It was easy in this case for Cantor to conclude that a number $\eta$ could be taken in this interval which was not listed in (12). Clearly any real number $\eta \in (\alpha^{(\nu)}, \beta^{(\nu)})$ would suffice, as long as $\eta$ was not the one element possible listed in (12).

In the case when the number of intervals $(\alpha^{(\nu)}, \beta^{(\nu)})$ were not finite, Cantor's argument shifted to consider two alternatives in the limit. Since the progressing sequence $\alpha, \alpha', ..., \alpha^{(\nu)}, ...$ did not increase indefinitely, but was bounded within $(\alpha, \beta)$, it had to assume an upper limit which Cantor denoted $\alpha^\infty$. Similarly, the regression sequence $\beta, \beta', ..., \beta^{(\nu)}, ...$ was assigned the lower limit $\beta^\infty$. Where $\alpha^\infty < \beta^\infty$, then, as in the finite case, any real number $\eta \in (\alpha^\infty, \beta^\infty)$ was sufficient to produce the necessary real number not listed in (12). However, were $\alpha^\infty = \beta^\infty$, Cantor reasoned that $\eta = \alpha^\infty = \beta^\infty$ could not be included as an element of (12) (we will prove that this assumption is not correct). He designed $\eta = \omega_\rho$. But $\omega_\rho$, for sufficiently large index $\nu$, would be excluded from all intervals nested within $(\alpha^{(\nu)}, \beta^{(\nu)})$. Nevertheless, by virtue of the construction Cantor had given, $\eta$ had to lie in every interval $(\alpha^{(\nu)}, \beta^{(\nu)})$, regardless of index. The contradiction established the proof: $R$ was nondenumerable.

The main part of the proof is that there is a progression of elements $\alpha^{(n)}$ and regression of elements $\beta^{(n)}$, such that

$$\alpha < \alpha^{(1)} < \alpha^{(2)} < ... < ... < \beta^{(2)} < \beta^{(1)} < \beta \tag{13}$$

The progression ought to have an upper limit; but there is no element $\alpha^{(n)}$ which can serve as this upper limit, for if any element $\alpha^{(n)}$ is proposed, one can clearly carry the process just indicated that $\alpha^{(n)}$ will be outside the interval $\alpha^{(n)}...\beta^{(n)}$.

The best way to illustrate what is wrong with this proof is to apply it on the set of all rational numbers. Applying exactly the same procedure proposed by Cantor on the set of rational numbers from interval (0,2) it is possible to make the sequences that determine progression of elements $\alpha^{(n)}$, and regression of elements $\beta^{(n)}$ such that

$$0 < \frac{2}{4} < \frac{4}{6} < ... < \frac{2+2n}{4+2n} < ... < ... < \frac{4+2n}{2+2n} < ... < \frac{8}{6} < \frac{6}{4} < 2 \tag{14}$$

The above progression and regression of elements determine as the limit number $\eta = 1$. There is no element $\alpha^{(n)}$ from (14) which can serve as this upper limit. Following the Cantor's line of conclusion, for again $\eta = \alpha^\infty \in (\alpha^{(n)}, \beta^{(n)})$ for all $n$, and hence $\alpha^\infty \neq \alpha_n$ for all n, simply by the way the progression and regression sequences are constructed the number $\eta = 1$, which is obtained in the limit, cannot be in the sequence of rational numbers on interval $(0, 2)$. Therefore we should conclude that the set of rational numbers is not denumerable, while we know that it is denumerable. Why we get this contradiction?

Answer is simple. The above example demonstrates that the Cantor statement that the number $\eta$ which is obtained as the limit of progression and regression sequences cannot be an element of (12) is not correct. He stated that whatever

$\omega_\rho$ is taken for $\eta$ that for sufficient large index $\nu$, it will be excluded from all intervals nested within $(\alpha^{(\nu)}, \beta^{(\nu)})$. Obviously there is no element in sequences (13) or (14) which can serve as the limit $\eta$. Any number $\alpha^{(n)}$ taken from (13) or (14) will fail as Cantor properly stated. However, it is not correct that there is no number from (12), which is equal to $\eta$ and which will be inside any interval $(\alpha^{(\nu)}, \beta^{(\nu)})$ for any $\nu$. In our example number 1 is obviously in (12) and it is in all intervals of (13) or (14) regardless of how large is the $\nu$. So, creating nested intervals $(\alpha^{(\nu)}, \beta^{(\nu)})$ by following Cantor's procedure, as the result of regressing and progressing sequences, obtained is in the limit number $\eta$, which by his statement cannot be part of (12), because of the way how it is created. This is obviously not correct, since as our example demonstrates, the number $\eta = 1$ which is obtained following Cantor's procedure is obviously part of the sequence (12), since in our example (12) represents sequence of rational numbers and $\eta = 1$ is the part of that sequence. By getting for the limit in sequence (13) a number which is not an element of (13) it does not mean that the set of all real numbers is not countable. It only means that the particular sequence does not contain that number, but the same number may be an element of the sequence (12). The particular sequence (13) is not the only sequence that can be constructed from (12), so it does not need to contain all numbers from (12).

**Theorem 5**

By the Cantor's diagonal procedure, it is not possible to build numbers that are different from all numbers in a general assumed denumerable sequence of all real numbers. The numbers created on the diagonal of the assumed sequence have the values that are not different from the values of the numbers in the assumed denumerable sequence.

**Proof of the theorem 5**

In his proof Cantor first produced a countable listing of elements $E_\nu$ in terms of the corresponding array (15), where each $a_{\mu,\nu}$ was either m or w:

$$\begin{aligned} &E_1 = (a_{11}, a_{12}, ..., a_{1\nu}, ...) \\ &E_2 = (a_{21}, a_{22}, ..., a_{2\nu}, ...) \\ &\vdots \\ &E_\nu = (a_{\mu 1}, a_{\mu 2}, ..., a_{\mu\nu}, ...) \\ &\vdots \end{aligned} \tag{15}$$

Then he defined a new sequence $b_1, b_2, ..., b_\nu, ...$ . Each $b_\nu$ was either m or w,

determined so that $b_\nu \neq a_{\nu\nu}$. By formulating from this sequence of $b_\nu$ the element $E_0 = (b_1, b_2, ..., b_\nu, ...)$, it followed that $E_0 \neq E_\nu$ for any value of the index $\nu$.

Let apply Cantor's procedure on the set of real numbers from the interval (0,1) to answer what has in reality Cantor proved by his diagonal procedure. He claimed that it is possible on the diagonal of an arbitrary denumerable sequence, which represents numbers in the interval (0,1), to create numbers that are different from the first number in the first decimal point, that are different from the second number in the second decimal point, and so on. From that he concluded that the created numbers are not in that assumed sequence and that the real numbers are not denumerable. The idea of his proof is that in the arbitrary assumed denumerable set of real numbers, each element of the set, each real number, has to be related by a one-to one correspondence to a natural number, which has a final value. That is, any real number has to take a finite place in that sequence. After that he concluded that numbers on the diagonal will be different from any of numbers in the sequence, because they are different from the first, the second and other numbers. But has Cantor really proved this, and what does it means that a created number on the diagonal is different from the first number in the sequence, the second and so-on? By the way how a diagonal number is created it is obvious that it is different from the first number in the sequence in $10^{-1}$ of the magnitude, from the second on $10^{-2}$ and from an $n^{th}$ number on $10^{-n}$ order of the magnitude. Of course, it will be different from any $n$ elements in that sequence. From this and the earlier statement that any real number has to be assigned to a finite position in the sequence, Cantor concluded that the numbers created on the diagonal are different from all numbers in the arbitrary assumed sequence because any number has to be assigned to an $n$ that has a final value. What Cantor has proved by the diagonal procedure is that if we take any finite subset of the real set, i.e. first $n$ elements, it is possible to create an $n+1$ element by the defined diagonal procedure. It is true that with this procedure we can go further and build more and more elements of the real set, which are different from an finite subset that contains $n$ elements. However, he did not prove that it is possible to create new numbers that are not already included in the arbitrary assumed countable set, the numbers that will be different from all elements in that set. The conclusion that created diagonal elements are such numbers is not correct. A number created on the diagonal has to be different not only from the first $n$ elements in the set, regardless of how large is the $n$. The created number must be different from all numbers in that set, which as we know has an infinite number of elements. This is the main point, that the assumed denumerable sequence has an infinite, not finite, number of elements. However, by the way how the diagonal numbers are created it is obvious that they are different just from final subset from the assumed denumerable set and not from all numbers in that set. To prove that let us look what is the difference in the value between the numbers created on the diagonal and the

numbers included in the assumed denumerable set. The difference between these diagonal numbers and all other numbers in the assumed sequence is simply given by the equation:

$$\lim_{n\to\infty} 10^{-n} = 0 \tag{16}$$

In the above equation we have to take the limit when $n$ is going to infinity, since we have to take into account that the proposed denumerable set has infinite number of elements. The numbers created on the diagonal must be different from all these numbers, not only from a final size subset of these elements.

This proves that the numbers created on the diagonal are not different from all the numbers in that assumed denumerable set. They are different from a subset with $n$ elements, but they are contained in the proposed denumerable set. Cantor did not took into account that the proposed denumerable set has infinite number of elements. In his discussion he was always focused on some finite size subset, considering some finite number of elements that belong to some finite $n$. It is impossible by the proposed diagonal procedure to build numbers that are not included in the assumed denumerable set and particularly it is not possible by this way to create an ascending hierarchy, in fact a limitless sequence of transfinite powers.

## 4 Conclusion

It is shown that the set of all real numbers is denumerable. The general element that generates the set is given and the first few elements of the sequence that contains all real numbers are written explicitly. By three independent proofs it is shown that the proposed sequence represents the set of the numbers which is dense anywhere, that does not have any gaps, and that is similar to the set of all real numbers, which proves that the sequence contains all real numbers. It is also proven that the Cantor's 1873 proof of non denumerability is not correct since it implicates non denumerability of rational numbers. In addition it is proven that the numbers generated by the diagonal procedure in Cantor's 1991 proof are not different from the numbers in the assumed denumerable set.

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